QUASI-ISOLATED ELEMENTS IN REDUCTIVE GROUPS

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Abstract. A semisimple element s of a connected reductive group $\mathbf{G}$ is said *quasi-isolated* (respectively *isolated*) if $C_{\mathbf{G}}(s)$ (respectively $C^\circ_{\mathbf{G}}(s)$) is not contained in a Levi subgroup of a proper parabolic subgroup of $\mathbf{G}$. We study properties of quasi-isolated semisimple elements and give a classification in terms of the affine Dynkin diagram of $\mathbf{G}$. Tables are provided for adjoint simple groups.

1. Preliminaries and notation

1.A. Notation. Let $\mathbb{F}$ be an algebraically closed field. Let p denote its characteristic. By a variety (respectively an algebraic group), we mean an algebraic variety (respectively an algebraic group) over $\mathbb{F}$. We denote by $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$ (in particular, if $p = 0$, then $\mathbb{Z}_{(p)} = \mathbb{Q}$).

We fix a connected reductive group $\mathbf{G}$. We denote by $\mathbf{Z}(\mathbf{G})$ its center and $\mathbf{D}(\mathbf{G})$ its derived subgroup. If $g \in \mathbf{G}$, we denote by g_s (respectively g_u) its semisimple (respectively unipotent) part, $C_{\mathbf{G}}(g)$ its centralizer and $C^\circ_{\mathbf{G}}(g)$ the neutral component of $C_{\mathbf{G}}(g)$. We denote by $o(g) \in \{1, 2, 3, \dots\} \cup \{\infty\}$ the order of g.

1.B. Isolated and quasi-isolated elements. The element $g \in \mathbf{G}$ is said *quasi-isolated* (respectively *isolated*) if $C_{\mathbf{G}}(g_s)$ (respectively $C^\circ_{\mathbf{G}}(g_s)$) is not contained in a Levi subgroup of a proper parabolic subgroup of $\mathbf{G}$. If there is some ambiguity, we will speak about $\mathbf{G}$-isolated or $\mathbf{G}$-quasi-isolated elements to refer to the ambient group. Of course, an isolated element is quasi-isolated.

The isolated elements are present in many different papers while the quasi-isolated ones are not often mentionned (see [Bon, §4.5]). One reason might be the following : if the derived group of $\mathbf{G}$ is simply connected, then centralizers of semisimple elements are connected (by a theorem of Steinberg [S, Theorem 8.1], see also [Bou, Chapter VI, §2, Exercise 1]) so the notions of isolated and quasi-isolated elements coincide. Another possible reason is that the notion of isolated element depends only on the Dynkin diagram of $\mathbf{G}$, by opposition to the notion of quasi-isolated element (see Proposition 2.3 and Example 2.4).

Whenever the derived group of $\mathbf{G}$ is not simply connected, a quasi-isolated element might not be isolated. The following extreme case can even happen : there exist quasi-isolated semisimple elements s which are *regular* (that is, such that $C^\circ_{\mathbf{G}}(s)$ is a maximal torus), as it is shown by the following example.

Example 1.1 - Let $n \geqslant 2$ be a natural number invertible in $\mathbb{F}$. Let us assume in this example that $\mathbf{G} = \mathbf{PGL}_n(\mathbb{F})$. Let ζ be a primitive n-th root of unity in $\mathbb{F}$ and let s be the image of $\mathrm{diag}(1, \zeta, \zeta^2, \dots, \zeta^{n-1}) \in \mathbf{GL}_n(\mathbb{F})$ in $\mathbf{G}$. Then $C^\circ_{\mathbf{G}}(s)$ is the maximal torus consisting of the image of diagonal matrices in $\mathbf{G}$ (in particular, s is regular, so it is not isolated) but $C_{\mathbf{G}}(s)/C^\circ_{\mathbf{G}}(s)$ is cyclic of order n : it is generated by a Coxeter element of the Weyl group of $\mathbf{G}$ relatively to $C^\circ_{\mathbf{G}}(s)$. Therefore, s is quasi-isolated. □

1.C. Root system. The notions of isolated and quasi-isolated elements involve only the semisimple part, so we will focus on semisimple elements. For this reason, we fix once and for all a maximal torus of

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$\mathbf{G}$: determining if an element of this torus is quasi-isolated or not can be done thanks to the root system or the Weyl group relatively to this torus.

Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ and let $\mathbf{T}$ be a maximal torus of $\mathbf{B}$. Let W be the Weyl group and let Φ be the root system of $\mathbf{G}$ relatively to $\mathbf{T}$. Let Φ^+ (respectively Δ) denote the positive root system (respectively the basis) of Φ associated to $\mathbf{B}$.

We fix once and for all an element $s \in \mathbf{T}$. We denote by $\Phi(s)$ and by $W^\circ(s)$ respectively the root system and the Weyl group of $C^\circ_{\mathbf{G}}(s)$ relatively to $\mathbf{T}$. We set :

$$W(s) = \{w \in W \mid {}^w s = s\}.$$

Let $\mathbf{B}(s)$ be a Borel subgroup of $C^\circ_{\mathbf{G}}(s)$ containing $\mathbf{T}$ and let $\Phi^+(s)$ (respectively $\Delta(s)$) denote the positive root system (respectively the basis) of $\Phi(s)$ associated to $\mathbf{B}(s)$. We set :

$$A(s) = \{w \in W(s) \mid w(\Phi^+(s)) = \Phi^+(s)\}.$$

EXAMPLE 1.2 - $C^\circ_{\mathbf{B}}(s)$ is a Borel subgroup of $C^\circ_{\mathbf{G}}(s)$ containing $\mathbf{T}$. If $\mathbf{B}(s) = C^\circ_{\mathbf{B}}(s)$, then $\Phi^+(s) = \Phi^+ \cap \Phi(s)$. $\square$

We gather some elementary facts :

Proposition 1.3. *Let $s \in \mathbf{T}$. Then :*

(a) $\Phi(s) = \{\alpha \in \Phi \mid \alpha(s) = 1\}$.
(b) *$W(s)$ is the Weyl group of $C_{\mathbf{G}}(s)$ relatively to $\mathbf{T}$.*
(c) $W(s) = A(s) \ltimes W^\circ(s)$.
(d) $A(s) \simeq C_{\mathbf{G}}(s)/C^\circ_{\mathbf{G}}(s)$.

Corollary 1.4. *Let $s \in \mathbf{T}$. Then :*

(a) *s is isolated (respectively quasi-isolated) if and only if $W^\circ(s)$ (respectively $W(s)$) is not contained in a proper parabolic subgroup of W.*
(b) *The following are equivalent :*
    (1) *s is isolated ;*
    (2) *$\Phi(s)$ is not contained in a proper parabolic subsystem of Φ ;*
    (c) $|\Delta(s)| = |\Delta|$.

Proposition 1.5. *Let $s \in \mathbf{T}$. Then there exists an element $s' \in \mathbf{T}$, of finite order, such that $C_{\mathbf{G}}(s) = C_{\mathbf{G}}(s')$.*

PROOF - Let $\mathbf{S}$ denote the Zarisky closure of the group generated by s. Then $\mathbf{S}/\mathbf{S}^\circ$ is generated by the image of s, so it is cyclic. Moreover, $C_{\mathbf{G}}(s) = C_{\mathbf{G}}(\mathbf{S})$. Therefore, Proposition 1.5 follows immediately from the following easy lemma :

Lemma 1.6. *Let $\mathbf{D}$ be a diagonalizable group acting on an affine variety $\mathbf{X}$. We assume that $\mathbf{D}/\mathbf{D}^\circ$ is cyclic. Then there exists an element $t \in \mathbf{D}$ of finite order such that $\mathbf{X}^{\mathbf{D}} = \mathbf{X}^t$.*

PROOF OF LEMMA 1.6 - We first prove the following statement :

$(*)$ *For every prime number $\ell \neq p$, there exists an element $t \in \mathbf{D}^\circ$ of ℓ-power order such that $\mathbf{X}^{\mathbf{D}^\circ} = \mathbf{X}^t$.*

Proof of $(*)$ - By [Bor, Proposition 1.12], there exists a finite dimensional rational representation V of $\mathbf{D}^\circ$ and a $\mathbf{D}^\circ$-equivariant closed embedding $\mathbf{X} \hookrightarrow V$. So $\mathbf{X}^{\mathbf{D}^\circ} = V^{\mathbf{D}^\circ} \cap \mathbf{X}$. If $\chi \in X(\mathbf{D}^\circ)$, we set
$$V_\chi = \{v \in V \mid \forall t \in \mathbf{D},\ t.v = \chi(t)v\}.$$
Let $\chi_1, \ldots, \chi_k \in X(\mathbf{D}^\circ)$ be the distinct non-zero weights of $\mathbf{D}^\circ$ in its action on V. Then :
$$V = V^{\mathbf{D}^\circ} \oplus \Bigl(\mathop{\oplus}_{i=1}^{k} V_{\chi_i}\Bigr).$$
Now, the subgroup L of $\mathbf{D}^\circ$ consisting of elements of ℓ-power order is Zarisky dense in $\mathbf{D}^\circ$. Therefore, there exists $t \in L$ such that $t \notin \operatorname{Ker}\chi_1 \cup \cdots \cup \operatorname{Ker}\chi_k$. This implies that $V^{\mathbf{D}^\circ} = V^t$. This shows $(*)$. □

Now let D be a finite cyclic subgroup of $\mathbf{D}$ such that $\mathbf{D} = D \times \mathbf{D}^\circ$. Let $d \in D$ be such that $D = < d >$. Then $\mathbf{X}^{\mathbf{D}} = (\mathbf{X}^{\mathbf{D}^\circ})^d$. Let ℓ be a prime number greater than $|D|$. Then, by $(*)$, there exists $t \in \mathbf{D}^\circ$ of ℓ-power order such that $\mathbf{X}^{\mathbf{D}^\circ} = \mathbf{X}^t$. Then, since t and d have coprime order, we have $\mathbf{X}^{td} = (\mathbf{X}^t)^d = (\mathbf{X}^{\mathbf{D}^\circ})^d = \mathbf{X}^{\mathbf{D}}$. ■

Remark - Lemma 1.6 and statement $(*)$ are slight refinements of a well-known lemma on the action of tori on affine varieties (see for instance [DM, Proposition 0.7]). Moreover, Lemma 1.6 does not remain valid if $\mathbf{D}/\mathbf{D}^\circ$ is not cyclic. For instance, assume that $p \neq 2$ and consider the action of $\mathbf{D} = \{1,-1\} \times \{1,-1\}$ on $\mathbf{A}^3(\mathbb{F})$ by $(\varepsilon,\varepsilon').(x,y,z) = (\varepsilon x, \varepsilon' y, \varepsilon\varepsilon' z)$. □

2. Isotypic morphisms

2.A. Definition. A morphism $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$ is said *isotypic* if $\tilde{\mathbf{G}}$ is a connected reductive group, if $\operatorname{Ker}\pi$ is central in $\tilde{\mathbf{G}}$ and if $\operatorname{Im}\pi$ contains the derived group of $\mathbf{G}$.

Example and notation - Let $\pi_{\mathrm{sc}} : \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ be a simply connected covering of the derived group of $\mathbf{G}$. Let $\mathbf{G}_{\mathrm{ad}}$ denote the adjoint group of $\mathbf{G}$ and let $\pi_{\mathrm{ad}} : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ be the canonical surjective morphism. Then π_{sc} and π_{ad} are isotypic morphisms. We set $\mathbf{B}_{\mathrm{ad}} = \pi_{\mathrm{ad}}(\mathbf{B})$ and $\mathbf{T}_{\mathrm{ad}} = \pi(\mathbf{T})$. Then $\mathbf{B}_{\mathrm{ad}}$ is a Borel subgroup of $\mathbf{G}_{\mathrm{ad}}$ and $\mathbf{T}_{\mathrm{ad}}$ is a maximal torus of $\mathbf{B}_{\mathrm{ad}}$. Moreover, if $t \in \mathbf{T}$, we set $\bar{t} = \pi_{\mathrm{ad}}(t) \in \mathbf{T}_{\mathrm{ad}}$. □

We fix in this section an isotypic morphism $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$. Let $\operatorname{Ker}'\pi = \mathbf{D}(\tilde{\mathbf{G}}) \cap \operatorname{Ker}\pi$. It must be noticed that $\operatorname{Ker}'\pi$ is a finite abelian group of order prime to p. Let $\tilde{\mathbf{B}} = \pi^{-1}(\mathbf{B})$ and $\tilde{\mathbf{T}} = \pi^{-1}(\mathbf{T})$. Then $\tilde{\mathbf{B}}$ is a Borel subgroup of $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{T}}$ is a maximal torus of $\tilde{\mathbf{B}}$. We will identify the Weyl group of $\tilde{\mathbf{G}}$ relatively to $\tilde{\mathbf{T}}$ with W through the morphism π. Let $\tilde{\Phi}$ denote the root system of $\tilde{\mathbf{G}}$ relatively to $\tilde{\mathbf{T}}$. Then the morphism $\pi^* : X(\mathbf{T}) \to X(\tilde{\mathbf{T}})$ induced by π provides a bijection $\Phi \xleftrightarrow{\sim} \tilde{\Phi}$.

Since $\operatorname{Im}\pi$ contains $\mathbf{D}(\mathbf{G})$, we have $\pi(\tilde{\mathbf{G}}).\mathbf{Z}(\mathbf{G})^\circ = \mathbf{G}$. We fix once and for all in this section an element $\tilde{s} \in \tilde{\mathbf{T}}$ such that $\pi(\tilde{s}) \in s\mathbf{Z}(\mathbf{G})$. Then

(2.1) $$\pi(C_{\tilde{\mathbf{G}}}(\tilde{s})).\mathbf{Z}(\mathbf{G})^\circ \subset C_{\mathbf{G}}(s).$$

Moreover, by Proposition 1.3 (a), we have

(2.2) $$\pi(C^\circ_{\tilde{\mathbf{G}}}(\tilde{s})).\mathbf{Z}(\mathbf{G})^\circ = C^\circ_{\mathbf{G}}(s).$$

Therefore, $W(\tilde{s}) \subset W(s)$ and $W^\circ(\tilde{s}) = W^\circ(s)$. Moreover, $A(\tilde{s}) \subset A(s)$ (if we choose $\tilde{\mathbf{B}}(\tilde{s}) = \pi^{-1}(\mathbf{B}(s))$). These remarks have the following consequences :

Proposition 2.3. *With the above notation, we have :*

(a) *If $\tilde{s}$ is quasi-isolated in $\tilde{\mathbf{G}}$, then s is quasi-isolated in $\mathbf{G}$.*

(b) *$\tilde{s}$ is isolated in $\tilde{\mathbf{G}}$ if and only if s is isolated in $\mathbf{G}$.*

The following example shows that the converse to statement (a) of Proposition 2.3 is not true in general.

EXAMPLE 2.4 - Keep here the hypothesis and notation of Example 1.1. Assume that $\tilde{\mathbf{G}} = \mathbf{GL}_n(\mathbb{F})$ and that $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$ is the canonical morphism. Let $\tilde{s} = \mathrm{diag}(1, \zeta, \zeta^2, \dots, \zeta^{n-1})$. Then $\tilde{s}$ is not quasi-isolated in $\tilde{\mathbf{G}}$ since $C_{\tilde{\mathbf{G}}}(\tilde{s})$ is a maximal torus. But $s = \pi(\tilde{s})$ is quasi-isolated in $\mathbf{G}$ as it is shown in Example 1.1. □

REMARK 2.5 - If π is injective, then the inclusion 2.1 is an equality. So s is quasi-isolated in $\mathbf{G}$ if and only if $\tilde{s}$ is quasi-isolated in $\tilde{\mathbf{G}}$. □

2.B. The groups $A(s)$ et $A(\tilde{s})$. We will compare here the groups $A(s)$ and $A(\tilde{s})$ in order to obtain general properties of the group $A(s)$. Most of the results of this subsection are well-known, particularly the Corollary 2.9 (see [S, lemme 9.2] and [BM, lemme 2.1]) but they are rarely stated in the whole generality of this subsection.

Let $\mathrm{Com}(\mathbf{G})$ denote the set of couples $(x, y) \in \mathbf{G} \times \mathbf{G}$ such that $xy = yx$. This is a closed subvariety of $\mathbf{G} \times \mathbf{G}$. If $(x, y) \in \mathrm{Com}(\mathbf{G})$, we denote by $\omega(x, y)$ the element $[\tilde{x}, \tilde{y}] = \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} \in \tilde{\mathbf{G}}$ where $\tilde{x} \in \tilde{\mathbf{G}}$ and $\tilde{y}$ are two elements of $\tilde{\mathbf{G}}$ such that $\pi(\tilde{x}) \in x\mathbf{Z}(\mathbf{G})$ and $\pi(\tilde{y}) \in y\mathbf{Z}(\mathbf{G})$. It is easily checked that $\omega(x, y)$ depends only on x and y and does not depend on the choice of $\tilde{x}$ and $\tilde{y}$. Moreover, $\pi([\tilde{x}, \tilde{y}]) = [x, y] = 1$ so $\omega(x, y) \in \mathrm{Ker}' \pi$.

Lemma 2.6. *Let x, x', y and y' be four elements of $\mathbf{G}$ such that $xy = yx$, $x'y = yx'$ and $xy' = y'x$. Then :*

$$\omega(x', yy') = \omega(x, y)\omega(x, y'),$$
$$\omega(xx', y) = \omega(x, y)\omega(x', y)$$

and

$$\omega(x, y) = \omega(y, x)^{-1}.$$

PROOF - Let us show the first equality (the second can be shown similarly and the third one is obvious). Let $\tilde{x}$, $\tilde{y}$ and $\tilde{y}'$ be three elements of $\tilde{\mathbf{G}}$ such that $\pi(\tilde{x}) \in x\mathbf{Z}(\mathbf{G})$, $\pi(\tilde{y}) \in y\mathbf{Z}(\mathbf{G})$ and $\pi(\tilde{y}') \in y'\mathbf{Z}(\mathbf{G})$. Then $\pi(\tilde{y}\tilde{y}') \in yy'\mathbf{Z}(\mathbf{G})$ and so

$$\begin{aligned}
\omega(x, yy') &= [\tilde{x}, \tilde{y}\tilde{y}'] \\
&= \tilde{x}\tilde{y}\tilde{y}'\tilde{x}^{-1}\tilde{y}'^{-1}\tilde{y}^{-1} \\
&= \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{x}\tilde{y}'\tilde{x}^{-1}\tilde{y}'^{-1}\tilde{y}^{-1} \\
&= \tilde{x}\tilde{y}\tilde{x}^{-1}\omega(x, y')\tilde{y}^{-1} \\
&= \omega(x, y)\omega(x, y'),
\end{aligned}$$

where the last equality follows from the fact that $\omega(x, y')$ is central in $\tilde{\mathbf{G}}$. ■

Let $\omega_s : C_{\mathbf{G}}(s) \to \mathrm{Ker}' \pi$, $g \mapsto \omega(g, s)$. The Lemma 2.6 shows that ω_s is a morphism of groups.

Lemma 2.7. $\mathrm{Ker}\, \omega_s = \pi(C_{\tilde{\mathbf{G}}}(\tilde{s})).\mathbf{Z}(\mathbf{G})^\circ$.

PROOF - Let $g \in \mathrm{Ker}\, \omega_s$. There exists $\tilde{g} \in \tilde{\mathbf{G}}$ such that $\pi(\tilde{g}) \in g\mathbf{Z}(\mathbf{G})^\circ$. Since $\omega_s(g) = [\tilde{g}, \tilde{s}] = 1$, we have $\tilde{g} \in C_{\tilde{\mathbf{G}}}(\tilde{s})$. ■

Corollary 2.8. *We have :*

(a) *ω_s induces a morphism $\tilde{\omega}_s : A(s) \longrightarrow \mathrm{Ker}' \pi$. We have $\mathrm{Ker}\, \tilde{\omega}_s = A(\tilde{s})$ and $\mathrm{Im}\, \tilde{\omega}_s = \mathrm{Im}\, \omega_s = \{z \in \mathrm{Ker}\, \pi \mid \tilde{s}$ and $\tilde{s}z$ are conjugated in $\tilde{\mathbf{G}}\}$.*

(b) *$|A(s)/A(\tilde{s})|$ is a finite abelian group of order dividing $|\mathrm{Ker}' \pi|$ (so prime to p).*

PROOF - (a) By Lemma 2.7 and equality 2.2, $C^\circ_{\mathbf{G}}(s)$ is contained in the kernel of ω_s. Since $A(s) \simeq C_{\mathbf{G}}(s)/C^\circ_{\mathbf{G}}(s)$, we get the first assertion. The second follows again from Lemma 2.7.

Let us show the last one. Let $A = \{z \in \operatorname{Ker}\pi \mid \tilde{s} \text{ and } \tilde{s}z \text{ are conjugated in } \tilde{\mathbf{G}}\}$. Let $g \in C_{\mathbf{G}}(s)$. Let $\tilde{g}$ be an element of $\tilde{\mathbf{G}}$ such that $\pi(\tilde{g}) \in g\mathbf{Z}(\mathbf{G})$. Set $z = \omega_s(g)$. Then $\tilde{s}z = \tilde{g}\tilde{s}\tilde{g}^{-1}$, which shows that $z \in A$. So $\operatorname{Im}\omega_s \subset A$. Conversely, let $z \in A$. Then there exists $\tilde{g} \in \tilde{\mathbf{G}}$ such that $\tilde{s}z = \tilde{g}\tilde{s}\tilde{g}^{-1}$. Set $g = \pi(\tilde{g})$. Then $z = [\tilde{g}, \tilde{s}]$, which shows that $g \in C_{\mathbf{G}}(s)$ and that $z = \omega_s(g)$. So $A \subset \operatorname{Im}\omega_s$.

(b) follows immediately from (a). ∎

Corollary 2.9. *The group $A(s)$ is isomorphic to a subgroup of the p'-part of the fundamental group of $\mathbf{D}(\mathbf{G})$. The exponent of $A(s)$ divides the order of $\bar{s}$ in $\mathbf{G}_{\mathrm{ad}}$, whenever this one is finite.*

PROOF - This statement does not involve the group $\tilde{\mathbf{G}}$. So we can choose for π the most convenient morphism for this question. We thus assume that $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$ is the morphism $\pi_{\mathrm{sc}} : \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ defined in §2.A. Then $\operatorname{Ker}'\pi = \operatorname{Ker}\pi$ is the p'-part of the fundamental group of $\mathbf{G}$.

Moreover, Steinberg's Theorem [S, théorème 8.1] tells us that $C_{\tilde{\mathbf{G}}}(\tilde{s})$ is connected, so $A(\tilde{s}) = 1$. So the first assertion follows immediately from Corollary 2.8 (b). Let us show now the second assertion. Let n denote the order of $\bar{s}$ in $\mathbf{G}_{\mathrm{ad}}$ and let $g \in C_{\mathbf{G}}(s)$. We must show that $g^n \in C^\circ_{\mathbf{G}}(s)$. Then, by Lemma 2.6, we have $\omega_s(g^n) = \omega(g,s)^n = \omega(g,s^n)$. But, $\tilde{s}^n$ is central in $\tilde{\mathbf{G}}$. Thus $\omega_s(g^n) = 1$, which shows that $g^n \in \pi(C_{\tilde{\mathbf{G}}}(\tilde{s}))\mathbf{Z}(\mathbf{G})^\circ$ by Lemma 2.7. But, again by Steinberg's Theorem, we have $C_{\tilde{\mathbf{G}}}(\tilde{s}) = C^\circ_{\tilde{\mathbf{G}}}(\tilde{s})$, so $g^n \in C^\circ_{\mathbf{G}}(s)$ by 2.2. ∎

2.C. Isotypic morphisms and quasi-isolated elements. The Proposition 2.3 shows that the notion of isolated element depends only on the isogeny class $\mathbf{G}$. On the other hand, the Example 2.4 shows that the notion of quasi-isolated element does not behave so nicely. We will use the morphism ω_s to study a weak converse to the statement (a) of Proposition 2.3. This weak converse will also be used to obtain some classification result for quasi-isolated elements.

Let e_s^π denote the exponent of the group $A(s)/A(\tilde{s})$ (recall that e_s^π divides the exponent of $\operatorname{Ker}'\pi$ and the order of $\bar{s}$ in $\mathbf{G}_{\mathrm{ad}}$). A result analogous to the following has been shown in [Bon, preuve du corollaire 4.5.3].

Proposition 2.10. *The group $C_{\mathbf{G}}(s)$ is contained in $\pi(C_{\tilde{\mathbf{G}}}(\tilde{s}^{e_s^\pi})).\mathbf{Z}(\mathbf{G})^\circ$.*

PROOF - Let $g \in C_{\mathbf{G}}(s)$. Then $\omega_s(g)^{e_s^\pi} = 1$. But, by Lemma 2.6, we have $\omega_s(g)^{e_s^\pi} = \omega_{s^{e_s^\pi}}(g)$. This shows that $g \in \operatorname{Ker}\omega_{s^{e_s^\pi}} = \pi(C_{\tilde{\mathbf{G}}}(\tilde{s}^{e_s^\pi})).\mathbf{Z}(\mathbf{G})^\circ$ (see Lemma 2.7). ∎

Corollary 2.11. *If s is quasi-isolated in $\mathbf{G}$, then $\tilde{s}^{e_s^\pi}$ is quasi-isolated in $\tilde{\mathbf{G}}$.*

Corollary 2.12. *Let e be the exponent of $\operatorname{Ker}\pi_{\mathrm{sc}}$. If s is quasi-isolated in $\mathbf{G}$, then s^e is isolated in $\mathbf{G}$.*

PROOF - Once again, the group $\tilde{\mathbf{G}}$ is not involved in this statement, so we can assume here that $\pi = \pi_{\mathrm{sc}}$. Then e_s^π divides e so, by Corollary 2.11, $\tilde{s}^e$ is quasi-isolated in $\tilde{\mathbf{G}} = \mathbf{G}_{\mathrm{sc}}$. But, since $\tilde{\mathbf{G}}$ is simply connected, $\tilde{s}^e$ is isolated in $\tilde{\mathbf{G}}$. Therefore, by Proposition 1.4 (a), s^e is isolated in $\mathbf{G}$. ∎

Corollary 2.13. *If s is quasi-isolated, then $\bar{s}$ has finite order.*

PROOF - By Corollary 2.12, we may assume that s is isolated in $\mathbf{G}$. Let $\mu : X(\mathbf{T}_{\mathrm{ad}}) \to \mathbb{F}^\times$, $\chi \mapsto \chi(\bar{s})$. Then μ is a morphism of groups and $\Phi(s) = \Phi \cap \operatorname{Ker}\mu$ (here, we identify $X(\mathbf{T}_{\mathrm{ad}})$ to a subgroup of $X(\mathbf{T})$ via the morphism π_{ad}). Since s is isolated, it follows from Corollary 1.4 (b) that $< \Phi(s) >$ has finite index in $X(\mathbf{T}_{\mathrm{ad}})$. So $\operatorname{Ker}\mu$ has finite index in $X(\mathbf{T}_{\mathrm{ad}})$. Let d denote this index (that is the order of the image of μ). Then $\bar{s}$ has order d in $\mathbf{T}_{\mathrm{ad}}$. ■

3. Semisimple elements of finite order

We will describe in this subsection the possible structure of the centralizer of a semisimple element in $\mathbf{G}$. By Proposition 1.5, we can focus on semisimple elements of finite order. For this, we fix an injective morphism $\imath : (\mathbb{Q}/\mathbb{Z})_{p'} \hookrightarrow \mathbb{F}^\times$ and we denote by $\tilde{\imath} : \mathbb{Q} \to \mathbb{F}^\times$ the composition of the morphisms $\mathbb{Q} \longrightarrow (\mathbb{Q}/\mathbb{Z})_{p'} \xrightarrow{\imath} \mathbb{F}^\times$. Finally, we set $\tilde{\imath}_{\mathbf{T}} : \mathbb{Q} \otimes_{\mathbb{Z}} Y(\mathbf{T}) \longrightarrow \mathbf{T}$, $r \otimes_{\mathbb{Z}} \lambda \longmapsto \lambda(\tilde{\imath}(r))$. The image of $\tilde{\imath}_{\mathbf{T}}$ is the torsion subgroup of $\mathbf{T}$.

To understand the structure of $C_{\mathbf{G}}(s)$, then, by Proposition 1.5 and by Remark 2.5, it is sufficient to work under the following hypothesis :

> **Hypothesis -** *From now on, and until the end of this paper, we assume that* $\mathbf{G}$ *is semisimple and that* s *has finite order.*

REMARQUE - It must be noticed that, in view of classifying quasi-isolated semisimple elements, this hypothesis is not restrictive (see Remark 2.5 and Corollary 2.13). □

3.A. Preliminaries. Let V be the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} Y(\mathbf{T})$ and let V^* be its dual, identified with $\mathbb{Q} \otimes_{\mathbb{Z}} X(\mathbf{T})$. We denote by $<,>: V^* \times V \to \mathbb{Q}$ the canonical perfect pairing between V and V^*. Then $Y(\mathbf{T}_{\mathrm{sc}})$ may be identified with $< \Phi^\vee >$ and $X(\mathbf{T}_{\mathrm{ad}})$ may be identified with $< \Phi >$. Since $\mathbf{G}$ is semisimple, Δ is a basis of V^*. Let $(\varpi_\alpha^\vee)_{\alpha\in\Delta}$ be its dual basis. Then $Y(\mathbf{T}_{\mathrm{ad}})$ may be identified with $\oplus_{\alpha\in\Delta}\mathbb{Z}\varpi_\alpha^\vee$. As expected, we have $Y(\mathbf{T}_{\mathrm{sc}}) \subset Y(\mathbf{T}) \subset Y(\mathbf{T}_{\mathrm{ad}}) \subset V = \mathbb{Q} \otimes_{\mathbb{Z}} Y(\mathbf{T}_{\mathrm{sc}})$. If $v \in V$, let $\tau_v : V \to V$, $x \mapsto x + v$ denote the translation by v.

Let us recall the following elementary fact :

Lemma 3.1. *The map* $Y(\mathbf{T}_{\mathrm{ad}}) \to \mathbf{T}$, $\lambda \mapsto \tilde{\imath}_{\mathbf{T}}(\lambda)$ *induces an isomorphism* $(Y(\mathbf{T}_{\mathrm{ad}})/Y(\mathbf{T}))_{p'} \simeq \mathbf{Z}(\mathbf{G})$. *The map* $Y(\mathbf{T}) \to \mathbf{T}_{\mathrm{sc}}$, $\lambda \mapsto \tilde{\imath}_{\mathbf{T}_{\mathrm{sc}}}(\lambda)$ *induces an isomorphism* $(Y(\mathbf{T})/Y(\mathbf{T}_{\mathrm{sc}}))_{p'} \simeq \operatorname{Ker}\pi_{\mathrm{sc}}$.

If $\lambda \in V$, we set
$$\Phi(\lambda) = \{\alpha \in \Phi \mid \ < \alpha, \lambda > \in \mathbb{Z}\}$$
and
$$W_{\mathbf{G}}(\lambda) = \{w \in W \mid w(\lambda) - \lambda \in Y(\mathbf{T})\}.$$
We denote by $o_{\mathrm{sc}}(\lambda)$ (respectively $o_{\mathrm{ad}}(\lambda)$, respectively $o_{\mathbf{G}}(\lambda)$) the order of the image of λ in $V/Y(\mathbf{T}_{\mathrm{sc}})$ (respectively $V/Y(\mathbf{T}_{\mathrm{ad}})$, respectively $V/Y(\mathbf{T})$). Let $W^\circ(\lambda)$ denote the Weyl group of the closed subsystem $\Phi(\lambda)$ of Φ. Then $W^\circ(\lambda)$ is a normal subgroup of $W_{\mathbf{G}}(\lambda)$. If we fix a positive root system $\Phi^+(\lambda)$ in $\Phi(\lambda)$, then we can define
$$A_{\mathbf{G}}(\lambda) = \{w \in W_{\mathbf{G}}(\lambda) \mid w(\Phi^+(\lambda)) = \Phi^+(\lambda)\}.$$
Then
$$W_{\mathbf{G}}(\lambda) = A_{\mathbf{G}}(\lambda) \ltimes W^\circ(\lambda).$$
The next lemma shows that, in order to understand the structure of $C_{\mathbf{G}}(s)$, it is necessary and sufficient to understand the structure of $W(\lambda)$, $W^\circ(\lambda)$ and $A_{\mathbf{G}}(\lambda)$.

Lemma 3.2. *Let* $\lambda \in \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} Y(\mathbf{T}_{\mathrm{sc}}) \subset V$ *and let* $s = \tilde{\imath}_{\mathbf{T}}(\lambda)$. *Then*

(a) $o_{\mathbf{G}}(\lambda)$ *is the order of* s *;*

(b) $\Phi(\lambda) = \Phi(s)$ *so* $W^\circ(\lambda) = W^\circ(s)$.
(c) $W_{\mathbf{G}}(\lambda) = W(s)$ *and, if* $\Phi^+(\lambda) = \Phi^+(s)$*, then* $A_{\mathbf{G}}(\lambda) = A(s)$.

By analogy, we say that λ is $\mathbf{G}$-*isolated* (respectively $\mathbf{G}$-*quasi-isolated*) if $W^\circ(\lambda)$ (respectively $W(\lambda)$) is not contained in a proper parabolic subgroup of W.

Let $W_{\mathrm{aff}} = W \ltimes Y(\mathbf{T}_{\mathrm{sc}})$ denote the affine Weyl group of Φ. If $\lambda \in V$, we set

$$W_{\mathrm{aff}}(\lambda) = \{w \in W_{\mathrm{aff}} \mid w(\lambda) = \lambda\}.$$

For the proof of the next proposition, see [DM, Lemme 13.14 and Remark 13.15 (i)] and [Bou, Chapter VI, §2, Exercise 1].

Proposition 3.3. *Let* $\lambda \in V$*. Then*

(a) $W_{\mathrm{aff}}(\lambda)$ *is generated by affine reflections. Its image in* W *is* $W^\circ(\lambda)$.
(b) $W^\circ(\lambda)$ *is the kernel of the map* $W_{\mathbf{G}}(\lambda) \to Y(\mathbf{T})/Y(\mathbf{T}_{\mathrm{sc}})$, $w \mapsto w(\lambda) - \lambda + Y(\mathbf{T}_{\mathrm{sc}})$.
(c) *The exponent of* $A_{\mathbf{G}}(\lambda)$ *divides* $o_{\mathbf{G}}(\lambda)$.

Remark - By Proposition 1.5, by Lemma 3.2 and by Proposition 3.3 we get that the centralizer of a semisimple element in a simply connected group is connected (Steinberg's Theorem). □

3.B. Affine Dynkin diagram. We recall here some results from [Bou, Chapter VI, §2] concerning the affine Dynkin diagram associated to a root system. We denote by Φ_1, Φ_2,..., Φ_r the distinct irreducible components of Φ.

Let us fix $i \in \{1, 2, \dots, r\}$. Let $V_i = \mathbb{Q} \otimes_{\mathbb{Z}} < \Phi_i >$. Let W_i denote the Weyl group of Φ_i. We set $\Delta_i = \Delta \cap \Phi_i$, $\Phi_i^+ = \Phi^+ \cap \Phi_i$. Then $V_i = \oplus_{\alpha \in \Delta_i} \mathbb{Q} \varpi_\alpha^\vee$. We denote by $\tilde{\alpha}_i$ the highest root of Φ_i (with respect to the height defined by Δ_i). We write

$$\tilde{\alpha}_i = \sum_{\alpha \in \Delta_i} n_\alpha \alpha,$$

where the n_α are non-zero natural numbers ($\alpha \in \Delta_i$). By convention, we set $\varpi^\vee_{-\tilde{\alpha}_i} = 0$, $n_{-\tilde{\alpha}_i} = 1$. Let $\tilde{\Delta}_i = \Delta \cup \{-\tilde{\alpha}_i\}$, $\Delta_{i,\min} = \{\alpha \in \Delta_i \mid n_\alpha = 1\}$ and $\tilde{\Delta}_{i,\min} = \Delta_{i,\min} \cup \{-\tilde{\alpha}_i\}$. If $\alpha \in \tilde{\Delta}_{i,\min}$, we denote by Φ_α the parabolic subsystem of Φ_i with basis $\Delta_i - \{\alpha\}$ (for instance, $\Phi_{-\tilde{\alpha}_i} = \Phi_i$) and we set $\Phi_\alpha^+ = \Phi_i^+ \cap \Phi_\alpha$. Let W_α denote the Weyl group of the root system Φ_α and w_α its unique element such that $w_\alpha(\Phi_\alpha^+) = -\Phi_\alpha^+$. We set $z_\alpha = w_\alpha w_{-\tilde{\alpha}} \in W_i$ (note that $z_{-\tilde{\alpha}} = 1$) and

$$\mathrm{Aut}_{W_i}(\tilde{\Delta}_i) = \{z \in W_i \mid z(\tilde{\Delta}_i) = \tilde{\Delta}_i\}.$$

By [Bou, chapter VI, §2, Proposition 6], we have

$$\mathrm{Aut}_{W_i}(\tilde{\Delta}_i) = \{z_\alpha \mid \alpha \in \tilde{\Delta}_{i,\min}\}. \tag{3.4}$$

If $\alpha \in \Delta_i$, we set $m_\alpha = 0$. We also set $m_{-\tilde{\alpha}_i} = -1$. Now, let $\mathcal{C}_i$ denote the alcove

$$\begin{aligned}\mathcal{C}_i &= \{\lambda \in V_i \mid \forall\, \alpha \in \tilde{\Delta}_i,\ < \alpha, \lambda > \geqslant m_\alpha\} \\ &= \{\lambda \in V_i \mid (\forall\, \alpha \in \Delta_i,\ < \alpha, \lambda > \geqslant 0) \text{ and } < \tilde{\alpha}_i, \lambda > \leqslant 1\}.\end{aligned}$$

Then $\mathcal{C}_i$ is a fundamental domain for the action of the affine Weyl group $W_{i,\mathrm{aff}} = W_i \ltimes < \Phi_i^\vee >$ on V_i. Moreover, $\mathcal{C}_i$ is a closed simplex with vertices $(\varpi_\alpha^\vee / n_\alpha)_{\alpha \in \tilde{\Delta}_i}$.

With the above notation, we have :

$$W = W_1 \times W_2 \times \cdots \times W_r,$$

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r,$$

$$Y(\mathbf{T}_{\mathrm{sc}}) = \mathop{\oplus}_{i=1}^{r} \Big(V_i \cap Y(\mathbf{T}_{\mathrm{sc}})\Big)$$

and

$$Y(\mathbf{T}_{\mathrm{ad}}) = \mathop{\oplus}_{i=1}^{r} \Big(V_i \cap Y(\mathbf{T}_{\mathrm{ad}})\Big).$$

We set $\tilde{\Delta} = \tilde{\Delta}_1 \cup \tilde{\Delta}_2 \cup \cdots \cup \tilde{\Delta}_r$. Now, let

$$\mathcal{A} = \{z \in W \mid z(\tilde{\Delta}) = \tilde{\Delta}\}.$$

In other words, $\mathcal{A}$ is the automorphism group of the affine Dynkin diagram of $\mathbf{G}$ induced by an element of W. We have

$$\mathcal{A} = \mathrm{Aut}_{W_1}(\tilde{\Delta}_1) \times \mathrm{Aut}_{W_2}(\tilde{\Delta}_2) \times \cdots \times \mathrm{Aut}_{W_r}(\tilde{\Delta}_r).$$

If $z = (z_{\alpha_1}, z_{\alpha_2}, \ldots, z_{\alpha_r}) \in \mathcal{A}$, with $\alpha_i \in \tilde{\Delta}_{i,\min}$, we set

$$\varpi^\vee(z) = \varpi_{\alpha_1}^\vee + \varpi_{\alpha_2^\vee} + \cdots + \varpi_{\alpha_r}^\vee.$$

Finally, let

$$\begin{array}{rcl} \mathcal{C} & = & \{\lambda \in V \mid \forall\, \alpha \in \tilde{\Delta},\ <\alpha,\lambda> \geqslant m_\alpha\} \\ & = & \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_r. \end{array}$$

Then $\mathcal{C}$ is a fundamental domain for the action of W_{aff} in V. Then, by [Bou, Chapter VI, §2], we have, for every $z \in \mathcal{A}$,

$$z(\mathcal{C}) + \varpi^\vee(z) = \mathcal{C} \tag{3.5}$$

and the map

$$\begin{array}{rccl} \varpi^\vee : & \mathcal{A} & \longrightarrow & Y(\mathbf{T}_{\mathrm{ad}})/Y(\mathbf{T}_{\mathrm{sc}}) \\ & z & \longmapsto & \varpi^\vee(z) + Y(\mathbf{T}_{\mathrm{sc}}) \end{array} \tag{3.6}$$

is an isomorphism of groups. If $z = (z_{\alpha_1}, z_{\alpha_2}, \ldots, z_{\alpha_r}) \in \mathcal{A}$, with $\alpha_i \in \tilde{\Delta}_{i,\min}$, and if $\alpha \in \tilde{\Delta}_i$, then

$$n_{z(\alpha)} = n_\alpha \tag{3.7}$$

and

$$z(\frac{1}{n_\alpha}\varpi_\alpha^\vee) + \varpi_{\alpha_i}^\vee = \frac{1}{n_\alpha}\varpi_{z(\alpha)}^\vee. \tag{3.8}$$

Since we will be working with the affine Weyl group of W_{aff}, it will be convenient to work with "affine coordinates". More precisely, if $\lambda \in V$, we will denote by $(\lambda_\alpha)_{\alpha\in\tilde{\Delta}}$ the unique family of rational numbers such that

(1) $\forall\ i \in \{1, 2, \ldots, r\},\ \displaystyle\sum_{\alpha\in\tilde{\Delta}_i} \lambda_\alpha = 1$;

(2) $\lambda = \displaystyle\sum_{\alpha\in\tilde{\Delta}} \frac{\lambda_\alpha}{n_\alpha}\varpi_\alpha^\vee$.

Note that $\lambda \in \mathcal{C}$ if and only if $\lambda_\alpha \geqslant 0$ for every $\alpha \in \tilde{\Delta}$. Then, we have, for every $\alpha \in \tilde{\Delta}$,

$$<\alpha,\lambda> = \frac{\lambda_\alpha}{n_\alpha} + m_\alpha. \tag{3.9}$$

Proof of 3.9 - Recall that m_α has been defined in §3.A. If $\alpha \in \Delta$, then $m_\alpha = 0$ and, by (3), $<\alpha,\lambda> = \lambda_\alpha/n_\alpha$. On the other hand, if $\alpha \in \tilde{\Delta} - \Delta$, then $m_\alpha = -1$ and there exists a unique $i \in \{1, 2, \ldots, r\}$ such that $\alpha = -\tilde{\alpha}_i$. Therefore, by (2) and (3), $<\alpha,\lambda> = -\sum_{\beta\in\Delta_i}\lambda_\beta = \lambda_\alpha - 1$. ■

Moreover, it follows from 3.8 that, for every $z \in \mathcal{A}$,

$$z(\lambda) + \varpi^\vee(z) = \sum_{\alpha\in\tilde{\Delta}} \frac{\lambda_{z^{-1}(\alpha)}}{n_\alpha}\varpi_\alpha^\vee \tag{3.10}$$

In other words, $(z(\lambda) + \varpi^\vee(z))_\alpha = \lambda_{z^{-1}(\alpha)}$.

3.C. Orbits under the action of $W \ltimes Y(\mathbf{T})$. Let $\mathcal{A}_{\mathbf{G}}$ be the subgroup of $\mathcal{A}$ defined to be the inverse image of $Y(\mathbf{T})/Y(\mathbf{T}_{\mathrm{sc}})$ under the isomorphism $\varpi^\vee$. Since $\mathcal{C}$ is a fundamental domain for the action of W_{aff}, it will be interesting to understand whenever two elements of $\mathcal{C}$ are in the same orbit under $W \ltimes Y(\mathbf{T})$. The answer is given by the following proposition.

Proposition 3.11. *Let λ and μ be two elements of $\mathcal{C}$ and let $w \in W$. If $\mu - w(\lambda) \in Y(\mathbf{T})$, then there exists $w^\circ \in W^\circ(\lambda)$ and $z \in \mathcal{A}_{\mathbf{G}}$ such that $w = zw^\circ$. Moreover, if d is a common multiple of $o(\lambda)$ and $o(\mu)$, then $z^d = 1$.*

Proof - Assume that $\mu - w(\lambda) \in Y(\mathbf{T})$. Then there exists $z \in \mathcal{A}_{\mathbf{G}}$ and $u \in Y(\mathbf{T}_{\mathrm{sc}})$ such that $w(\lambda) - \mu = -\varpi^\vee(z) + u$. But, $(\tau_{-u}w)(\lambda) = \mu - \varpi^\vee(z) \in \mathcal{C} - \varpi^\vee(z) = z(\mathcal{C})$ (see 3.5). Therefore, $(z^{-1}\tau_{-u}w)(\lambda) \in \mathcal{C}$. Since $\mathcal{C}$ is a fundamental domain for the action of W_{aff} on V and since $z^{-1}\tau_{-u}w \in W_{\mathrm{aff}}$, we deduce that $z^{-1}\tau_{-u}w(\lambda) = \lambda$. So, by Proposition 3.3 (a), $z^{-1}w \in W^\circ(\lambda)$, as expected.

For the last assertion, note that the hypothesis implies that $d(w(\lambda)-\mu) \in Y(\mathbf{T}_{\mathrm{sc}})$. Therefore, $d\varpi^\vee(z) \in Y(\mathbf{T}_{\mathrm{sc}})$. Since the map 3.6 is an isomorphism, we get that $z^d = 1$. ■

Corollary 3.12. *Let λ and μ be two elements of $\mathcal{C}$. Then the following assertions are equivalent :*

(1) *λ and μ are in the same $W \ltimes Y(\mathbf{T})$-orbit.*
(2) *There exists $z \in \mathcal{A}_{\mathbf{G}}$ such that $z(\lambda) - \mu \in Y(\mathbf{T})$.*

Proof - Clear. ■

3.D. The group $W_{\mathbf{G}}(\lambda)$. Let us now come back to the aim of this section, namely the description of the group $W_{\mathbf{G}}(\lambda)$. Since $\mathcal{C}$ is a fundamental domain for the action of W_{aff} in V, it is sufficient to understand the structure of $W_{\mathbf{G}}(\lambda)$ whenever $\lambda \in \mathcal{C}$.

Proposition 3.13. *Let $\lambda \in \mathcal{C}$. We set $I_\lambda = \{\alpha \in \tilde{\Delta} \mid \lambda_\alpha = 0\} = \{\alpha \in \tilde{\Delta} \mid \ <\alpha,\lambda>= m_\alpha\}$. Then :*

(a) *I_λ is a basis of $\Phi(\lambda)$.*
(b) *If $\Phi^+(\lambda)$ is the poitive root system of $\Phi(\lambda)$ associated to the basis I_λ, then*
$$A^{\mathbf{G}}(\lambda) = \{z \in \mathcal{A}_{\mathbf{G}} \mid \forall\ \alpha \in \tilde{\Delta},\ \lambda_{z(\alpha)} = \lambda_\alpha\}.$$

Proof - (a) For $\alpha \in \tilde{\Delta}$, let $H_\alpha = \{v \in V \mid \ <\alpha, v>= m_\alpha\}$. Then $(H_\alpha)_{\alpha\in\tilde{\Delta}}$ is the family of walls of the alcôve $\mathcal{C}$. Moreover, W_{aff} is generated by the affine reflections with respect to the walls of $\mathcal{C}$ which contains λ (see [Bou, ??]). Therefore, $W^\circ(\lambda)$ is generated by the reflections $(s_\alpha)_{\alpha\in I_\lambda}$. Since $<\alpha,\beta^\vee> \leqslant 0$ for every α, $\beta \in \tilde{\Delta}$, this implies that I_λ is a basis of $\Phi(\lambda)$.

(b) Let $A = \{z \in \mathcal{A}_{\mathbf{G}} \mid \forall\ \alpha \in \tilde{\Delta},\ \lambda_{z(\alpha)} = \lambda_\alpha\}$. Then A stabilizes I_λ by construction and, for every $z \in A$, $z(\lambda) - \lambda = \varpi^\vee(z)Y(\mathbf{T})$ by 3.10. So $A \subset A_{\mathbf{G}}(\lambda)$. Let us prove now the reverse inclusion.

First, let us prove that $A_{\mathbf{G}}(\lambda) \subset \mathcal{A}_{\mathbf{G}}$. Let $z \in A_{\mathbf{G}}(\lambda)$. By Proposition 3.11, there exists $a \in \mathcal{A}_{\mathbf{G}}$ and $w^\circ \in W^\circ(\lambda)$ such that $z = aw^\circ$. So $a \in W(\lambda)$ and $a(I_\lambda) \subset \tilde{\Delta}$. In particular, $a(I_\lambda) \subset \Phi(\lambda) \cap \tilde{\Delta}$. But $\Phi(\lambda) \cap \tilde{\Delta} = I_\lambda$ by (a). So $a(I_\lambda) = I_\lambda$. Moreover, $z(I_\lambda) = I_\lambda$ by definition of $A_{\mathbf{G}}(\lambda)$. So $w^\circ(I_\lambda) = I_\lambda$ and $w^\circ \in W^\circ(\lambda)$, which implies that $w^\circ = 1$, that is $z = a$. This shows that $z \in \mathcal{A}_{\mathbf{G}}$.

Now, by 3.10, we have
$$z(\lambda) - \lambda + \varpi^\vee(z) = \sum_{i=1}^{r}\Big(\sum_{\alpha\in\Delta_i} \frac{\lambda_{z^{-1}(\alpha)} - \lambda_\alpha}{n_\alpha}\varpi_\alpha^\vee\Big) \in Y(\mathbf{T}) \subset Y(\mathbf{T}_{\mathrm{ad}}).$$
Since z stabilizes I_λ, we have, for every $\alpha \in \Delta$,
$$\lambda_{z^{-1}(\alpha)}\lambda_\alpha = 0 \Rightarrow \lambda_{z^{-1}(\alpha)} = \lambda_\alpha = 0.$$
Moreover, $0 \leqslant \lambda_a \leqslant 1$. Therefore, $(\lambda_{z^{-1}(\alpha)} - \lambda_\alpha)/n_\alpha \in]-1/n_\alpha, 1/n_\alpha[$. Moreover, $(\varpi_\alpha^\vee)_{\alpha\in\Delta}$ is a $\mathbb{Z}$-basis of $Y(\mathbf{T}_{\mathrm{ad}})$. So $\lambda_{z^{-1}(\alpha)} = \lambda_\alpha$ for every $\alpha \in \Delta$. Then, by condition (1), $\lambda_{z^{-1}(\alpha)} = \lambda_\alpha$ for every $\alpha \in \tilde{\Delta}$. ■

Remark 3.14 - Keep the notation of Proposition 3.13. Then it may happen that $A_{\mathbf{G}}(\lambda)$ is strictly contained in the stabilizer of I_λ in $\mathcal{A}_{\mathbf{G}}$. Take for instance $\mathbf{G} = \mathbf{PGL}_2(\mathbb{F})$ and $\lambda = \varpi_\alpha^\vee/3$ where α is the unique simple root of $\mathbf{G}$. □

Remark 3.15 - If $\lambda \in \mathcal{C}$, note that $I_\lambda \cap \tilde{\Delta}_i \neq \tilde{\Delta}_i$ for every $i \in \{1,2,\dots,r\}$. $\square$

If $\lambda \in \mathcal{C}$, we will choose for $\Phi^+(\lambda)$ the positive root subsystem of $\Phi(\lambda)$ associated to the basis I_λ.

4. Classification of quasi-isolated elements

4.A. A characterization of quasi-isolated elements. If I is a subset of $\tilde{\Delta}$ such that $I \cap \tilde{\Delta}_i \neq \tilde{\Delta}_i$ for every $i \in \{1,2,\dots,r\}$, we denote by Φ_I the root subsystem of Φ with basis I and by W_I the Weyl group of Φ_I. It must be noticed that W_I is not necessarily a parabolic subgroup of W. The Proposition 3.13 shows that, whenever $\lambda \in \mathcal{C}$, $W_{\mathbf{G}}(\lambda) = A \ltimes W_{I_\lambda}$ for some subgroup A of $\mathcal{A}$ stabilizing I_λ. To determine if such a subgroup is contained or not in a proper parabolic subgroup of W, we need to determine the dimension of its fixed-points space. This is done in general in the next lemma.

Lemma 4.1. *Let I be a subset of $\tilde{\Delta}$ such that $I \cap \tilde{\Delta}_i \neq \tilde{\Delta}_i$ for every $i \in \{1,2,\dots,r\}$ and let A be a subgroup of $\mathcal{A}$ stabilizing I. Let r' denote the number of orbits of A in $\tilde{\Delta} - I$. Then* $\dim_{\mathbb{Q}} V^{A \ltimes W_I} = r' - r$.

Proof - By taking direct products, we may assume that Φ is irreducible or, in other words, that $r = 1$. Let $V_I = \mathbb{Q} \otimes_{\mathbb{Z}} < \Phi_I >$ and let E_I be the orthogonal of I in V. Then $V = V_I \oplus E_I$ and AW_I° stabilizes V_I and E_I. Moreover,

$$\{v \in V_I \mid \forall\, w \in W_I,\ w(v) = v\} = \{0\}$$

and W_I acts trivially on E_I. Consequently,

$$V^{A \ltimes W_I} = E_I^A.$$

Let $\mathbb{Q}[\tilde{\Delta} - I]$ denote the $\mathbb{Q}$-vector space with basis $(e_\alpha)_{\alpha \in \tilde{\Delta} - I}$. This is a permutation A-module. Let $f : \mathbb{Q}[\tilde{\Delta} - I] \to E_I$ the $\mathbb{Q}$-linear map sending e_α on the projection of α in E_I (for every $\alpha \in \tilde{\Delta} - I$). Then f is a morphism of $\mathbb{Q}A$-modules, whose kernel has dimension 1 (because $|\tilde{\Delta}| = \dim V + 1$).

Let $M = \{v \in \mathbb{Q}[\tilde{\Delta} - I] \mid \forall z \in A,\ z(v) = v\}$. Then

$$\dim_{\mathbb{Q}} E_I^A = \dim_{\mathbb{Q}} M - \dim_{\mathbb{Q}}(M \cap \operatorname{Ker} f).$$

Since $\dim_{\mathbb{Q}} M = r'$, we only need to show that A acts trivially on $\operatorname{Ker} f$. But $\sum_{\alpha \in \tilde{\Delta}} n_\alpha \alpha = 0$. So, by projection on E_I, we get that $\operatorname{Ker} f$ is generated by $\sum_{\alpha \in \tilde{\Delta} - I} n_\alpha e_\alpha$. By equality 3.7, this element is invariant under the action of A. This completes the proof of Lemma 4.1. $\blacksquare$

Corollary 4.2. *Let I be a subset of $\tilde{\Delta}$ such that $I \cap \tilde{\Delta}_i \neq \tilde{\Delta}_i$ for every $i \in \{1,2,\dots,r\}$ and let A be a subgroup of $\mathcal{A}$ stabilizing I. Then $A.W_I$ is not contained in a proper parabolic subgroup of W if and only if A acts transitively on $\tilde{\Delta}_i - I$ for every $i \in \{1,2,\dots,r\}$.*

Proof - This follows immediately from Proposition 4.1. $\blacksquare$

Corollary 4.3. *Let $\lambda \in \mathcal{C}$. Then :*

(a) *λ is $\mathbf{G}$-isolated if and only if $|\tilde{\Delta}_i - I_\lambda| = 1$ for every $i \in \{1,2,\dots,r\}$.*
(b) *λ is $\mathbf{G}$-quasi-isolated if and only if $A_{\mathbf{G}}(\lambda)$ acts transitively on $\tilde{\Delta}_i - I_\lambda$ for every $i \in \{1,2,\dots,r\}$.*

PROOF - This follows immediately from Proposition 3.13 and Corollary 4.2. ■

4.B. Classification of quasi-isolated elements in V. We are now ready to complete the classification of conjugacy classes of $\mathbf{G}$-quasi-isolated elements in V. Let $\mathcal{Q}(\mathbf{G})$ denote the set of subsets Ω of $\tilde{\Delta}$, such that, for every $i \in \{1, 2, \dots, r\}$, $\Omega \cap \tilde{\Delta}_i \neq \varnothing$ and the stabilizer of Ω_i in $\mathcal{A}_{\mathbf{G}}$ acts transitively on $\tilde{\Delta}_i$. If Ω is such a subset, we set

$$\lambda_\Omega = \sum_{i=1}^{r} \Big(\frac{1}{n_i(\Omega)|\Omega \cap \tilde{\Delta}_i|} \sum_{\alpha \in \Omega \cap \tilde{\Delta}_i} \varpi_\alpha^\vee \Big),$$

where $n_i(\Omega)$ is equal to n_α for every $\alpha \in \Omega \cap \tilde{\Delta}_i$ (see equality 3.7). Note that $\mathcal{A}_{\mathbf{G}}$ acts on $\mathcal{Q}(\mathbf{G})$. Moreover, by 3.8, we have, for every $z \in \mathcal{A}_{\mathbf{G}}$,

$$z(\lambda_\Omega) + \varpi^\vee(z) = \lambda_{z(\Omega)}. \tag{4.4}$$

Finally, we denote by $o_i^{\mathbf{G}}(\Omega)$ the number $o_{\mathbf{G}}(\varpi_\alpha^\vee)$ where $\alpha \in \Omega \cap \tilde{\Delta}_i$. Note that this number is constant on $\Omega \cap \tilde{\Delta}_i$. All the work done in this section shows that :

Theorem 4.5. *With the above notation, we have :*

(a) *The map $\mathcal{Q}(\mathbf{G}) \to \mathcal{C}$, $\Omega \mapsto \lambda_\Omega$ induces a bijection between the set of orbits of $\mathcal{A}_{\mathbf{G}}$ in $\mathcal{Q}(\mathbf{G})$ and the set of $W \ltimes Y(\mathbf{T})$-orbits of quasi-isolated elements in V.*

(b) *Let $\Omega \in \mathcal{Q}(\mathbf{G})$. Then :*
  - (α) $W^\circ(\lambda_\Omega) = W_{\tilde{\Delta}-\Omega}$ *;*
  - (β) $A_{\mathbf{G}}(\lambda_\Omega) = \{z \in \mathcal{A}_{\mathbf{G}} \mid z(\Omega) = \Omega$ *;*
  - (γ) $o_{\mathbf{G}}(\lambda)$ *is the lowest common multiple of* $(n_i(\Omega) o_i^{\mathbf{G}}(\Omega)|\Omega \cap \tilde{\Delta}_i|)_{1 \leqslant i \leqslant r}$ *;*
  - (δ) λ_Ω *is* $\mathbf{G}$*-isolated if and only if* $|\Omega_i| = 1$ *for every* $i \in \{1, 2, \dots, r\}$.

4.C. Classification of quasi-isolated semisimple elements. Let $\tilde{\Delta}_{p'}$ denote the subset of elements $\alpha \in \tilde{\Delta}$ such that $\varpi_\alpha^\vee / n_\alpha \in \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} Y(\mathbf{T}_{\mathrm{sc}})$. Let $\mathcal{Q}(\mathbf{G})_{p'}$ denote the set of $\Omega \in \mathcal{Q}(\mathbf{G})$ such that $\Omega \subset \tilde{\Delta}_{p'}$ and, for every $i \in \{1, 2, \dots, r\}$, p does not divide $|\Omega \cap \tilde{\Delta}_i|$. If $\Omega \in \mathcal{Q}(\mathbf{G})_{p'}$, we set $t_\Omega = \tilde{\imath}_{\mathbf{T}}(\lambda_\Omega) \in \mathbf{T}$.

Theorem 4.6. *With the above notation, we have :*

(a) *The map $\mathcal{Q}(\mathbf{G})_{p'} \to \mathbf{T}$, $\Omega \mapsto t_\Omega$ induces a bijection between the set of orbits of $(\mathcal{A}_{\mathbf{G}})_{p'}$ in $\mathcal{Q}(\mathbf{G})_{p'}$ and the set of conjugacy classes of quasi-isolated semisimple elements in $\mathbf{G}$.*

(b) *If $\Omega \in \mathcal{Q}(\mathbf{G})_{p'}$ then :*
  - (α) $W^\circ(t_\Omega) = W_{\tilde{\Delta}-\Omega}$ *;*
  - (β) $A_{\mathbf{G}}(t_\Omega) = \{z \in \mathcal{A}_{\mathbf{G}} \mid z(\Omega) = \Omega\}$ *;*
  - (γ) $o(t_\Omega)$ *is the lowest common multiple of* $(n_i(\Omega) o_i^{\mathbf{G}}(\Omega)|\Omega \cap \tilde{\Delta}_i|)_{1 \leqslant i \leqslant r}$ *;*
  - (δ) t_Ω *is* $\mathbf{G}$*-isolated if and only if* $|\Omega_i| = 1$ *for every* $i \in \{1, 2, \dots, r\}$.

PROOF - By Theorem 4.5 and Lemma 3.2, it is enough to show that the map $\mathcal{Q}(\mathbf{G})_{p'} \to \mathcal{C}$, $\Omega \mapsto \lambda_\Omega$ induces a bijection between the set of orbits of $(\mathcal{A}_{\mathbf{G}})_{p'}$ in $\mathcal{Q}(\mathbf{G})_{p'}$ to the set of $W \ltimes Y(\mathbf{T})$-orbits of quasi-isolated elements λ in V such that p does not divide $o_{\mathrm{sc}}(\lambda)$. But this follows from Theorem 4.5 (b) (γ) and the last assertion of Proposition 3.11. ■

REMARK 4.7 - We recall that the prime number p is said to be *very good* for $\mathbf{G}$ if it does not divide the numbers n_α ($\alpha \in \Delta$) and $|\mathcal{A}| = |Y(\mathbf{T}_{\mathrm{sc}})|/|Y(\mathbf{T}_{\mathrm{ad}})|$. We say here that p is *almost very good* for $\mathbf{G}$ is it does not divide the numbers n_α ($\alpha \in \tilde{\Delta}$) and $|\mathcal{A}_{\mathbf{G}}| = |Y(\mathbf{T})|/|Y(\mathbf{T}_{\mathrm{ad}})|$. If p is very good, then it is almost very good.

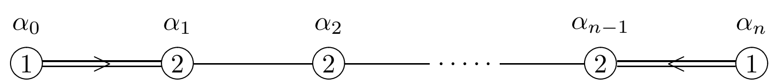
If p is almost very good, then $\tilde{\Delta}_{p'} = \tilde{\Delta}$ and $(\mathcal{A}_{\mathbf{G}})_{p'} = \mathcal{A}_{\mathbf{G}}$ so the set of $W \ltimes Y(\mathbf{T})$-orbits of $\mathbf{G}$-quasi-isolated elements in V is in natural bijection with the set of conjugacy classes of quasi-isolated semisimple elements in $\mathbf{G}$ (through the map $\tilde{\imath}_{\mathbf{T}}$). □

Example 4.8 - If all the irreducible components of Φ are of type B, C ou D and if $p = 2$, then $\tilde{\Delta}_{p'} = \{-\tilde{\alpha}_1, -\tilde{\alpha}_2, \dots, -\tilde{\alpha}_r\}$. Therefore, 1 is the unique quasi-isolated element in $\mathbf{G}$. □

4.D. Simply connected groups. If $\mathbf{G}$ is simply connected then $\mathcal{A}_{\mathbf{G}} = \{1\}$. Therefore, we retrieve the well-known classification of isolated semisimple elements in $\mathbf{G}$:

Proposition 4.9. *Assume that* $\mathbf{G}$ *is semisimple and simply connected. Then the map* $\tilde{\Delta}_{1,p'} \times \cdots \times \tilde{\Delta}_{r,p'} \to \mathbf{G}$, $(\alpha_1, \dots, \alpha_r) \mapsto \prod_{i=1}^r t_{\varpi_{\alpha_i}^\vee / n_{\alpha_i}}$ *induces a bijection between* $\tilde{\Delta}_{1,p'} \times \cdots \times \tilde{\Delta}_{r,p'}$ *and the set of conjugacy classes of (quasi-)isolated elements in* $\mathbf{G}$.

Example 4.10 - Assume here that $p \neq 2$ and that $\mathbf{G} = \mathbf{Sp}(V)$ where V is an even-dimensional vector space endowed with a non-degenerate alternating form. Let $\dim V = 2n$. Then $\mathbf{G}$ is simply connected, so $\mathcal{A}_{\mathbf{G}} = \{1\}$. Moreover, $\tilde{\Delta}_{p'} = \tilde{\Delta}$. Let us write $\alpha_0 = -\tilde{\alpha}_1$ and let us number the affine Dynkin diagram $\tilde{\Delta}$ of $\mathbf{G}$ as follows



The natural numbers written inside the node α_i is the number n_{α_i}. For $0 \leqslant i \leqslant n$, let $\Omega_i = \{\alpha_i\}$ and let $t_i = \tilde{\imath}_{\mathbf{T}}(\varpi_{\alpha_i}^\vee / n_{\alpha_i})$. Then $\{t_i \mid 0 \leqslant i \leqslant n\}$ is a set of representatives of conjugacy classes of isolated (i.e. quasi-isolated) elements in $\mathbf{G}$. Note that t_i is characterized by the following two properties :

$$t_i^2 = 1 \quad \text{and} \quad \dim \operatorname{Ker}(t_i + \mathrm{Id}_V) = i.$$

This shows that an element s is isolated in $\mathbf{G}$ if and only if $s^2 = 1$. Finally, note that $C_{\mathbf{G}}(t_i) = \mathbf{Sp}_{2i}(\mathbb{F}) \times \mathbf{Sp}_{2(n-i)}(\mathbb{F})$. ■

4.E. Special orthogonal groups. The case of special orthogonal groups in characteristic 2 has been treated in Example 4.8. In this subsection, we study the case of special orthogonal groups in good characteristic. We first adopt a naive point-of-view, using the natural representation of special orthogonal groups. At the end of this subsection, we will explain the link between this point-of-view and Theorem 4.6.

> **Hypothesis :** *Let us assume in this subsection, and only in this subsection, that* $p \neq 2$ *and that* $\mathbf{G} = \mathbf{SO}(V, \langle,\rangle) = \mathbf{SO}(V)$ *where* V *is a finite dimensional vector space over* $\mathbb{F}$ *and* $\langle,\rangle$ *is a non degenerate symmetric bilinear form on* V.

We denote by n the rank of $\mathbf{G}$. Then $n = \left[\dfrac{\dim V}{2}\right]$, except whenever $\dim V = 2$ (in this case, $n = 0$). If $s^2 = 1$, then $\dim \operatorname{Ker}(s + \mathrm{Id}_V) \equiv 0 \bmod 2$ so $\dim \operatorname{Ker}(s - \mathrm{Id}_V) \equiv \dim V \bmod 2$.

Proposition 4.11. *With this hypothesis, we have :*

(a) s *is quasi-isolated if and only if* $s^2 = 1$.
(b) *If* $s^2 = 1$, *then* s *is isolated if and only if* $\dim \operatorname{Ker}(s - \varepsilon\, \mathrm{Id}_V) \neq 1$ *for every* $\varepsilon \in \{1, -1\}$.

Proof - Assume first that there exists an eigenvalue ζ of s such that $\zeta^2 \neq 1$. Let V_ζ denote the ζ-eigenspace of s in V. Let E be the orthogonal subspace to $V_\zeta \oplus V_{\zeta^{-1}}$. We have

$$V = V_\zeta \oplus V_{\zeta^{-1}} \oplus E,$$

and this is an orthogonal decomposition. Therefore, the centralizer of s in $\mathbf{G}$ is contained in $\mathbf{G} \cap (\mathbf{GL}(V_\zeta) \times \mathbf{GL}(V_{\zeta^{-1}}) \times \mathbf{GL}(E))$, which is a Levi subgroup of a proper parabolic subgroup of $\mathbf{G}$. So s is not quasi-isolated.

Assume now that $s^2 = 1$. Then $V = V_1 \oplus V_{-1}$ and this decomposition is orthogonal. So

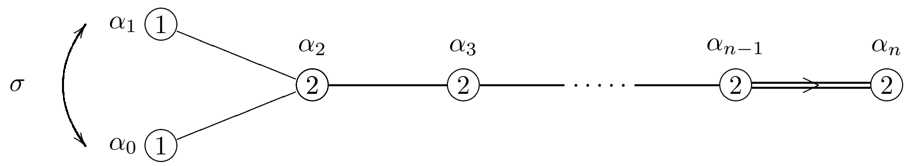
$$C_{\mathbf{G}}(s) = (\mathbf{O}(V_1) \times \mathbf{O}(V_{-1})) \cap \mathbf{G} \quad \text{et} \quad C^\circ_{\mathbf{G}}(s) = \mathbf{SO}(V_1) \times \mathbf{SO}(V_{-1}).$$

So s is quasi-isolated and it is isolated if and only if $\dim V_1 \neq 1$ and $\dim V_{-1} \neq 1$. ■

Corollary 4.14. *Keep the hypothesis of this subsection. If $0 \leqslant i \leqslant n$, let t_i denote a semisimple element of $\mathbf{G}$ such that $t_i^2 = 1$ and $\dim \operatorname{Ker}(t_i + \operatorname{Id}_V) = 2i$. Then $\{t_i \mid 0 \leqslant i \leqslant n\}$ is a set of representatives of conjugacy classes of quasi-isolated elements in $\mathbf{G}$. Moreover, t_i is isolated if and only if $i \notin \{1, (\dim V)/2-1\}$.*

Let us now compare the description given by Corollary 4.14 and the one given by Theorem 4.6. Since $p \neq 2$, we have $\tilde{\Delta}_{p'} = \tilde{\Delta}$ and $\mathcal{A}_{p'} = \mathcal{A}$ (indeed, p is very good for $\mathbf{G}$). For getting a uniform description, we assume that $\dim V \notin \{1, 2, 3, 4, 6\}$ (whenever $\dim V \in \{1, 2, 3, 4, 6\}$, then the reader can also check that Corollary 4.14 and Theorem 4.6 are still compatible !).

4.E.1. Type B. We assume here that $\dim V = 2n + 1$ and that $n \geqslant 2$. We set $\alpha_0 = -\tilde{\alpha}_1$. Then $\mathcal{A}_{\mathbf{G}} = \mathcal{A}$ is of order 2. We denote by σ its unique non-trivial element. We number the affine Dynkin diagram of $\mathbf{G}$ as follows :



The natural number written inside the node α_i is equal to n_{α_i}. We have

$$\sigma = {}^{s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{n-1}}} s_{\alpha_n}.$$

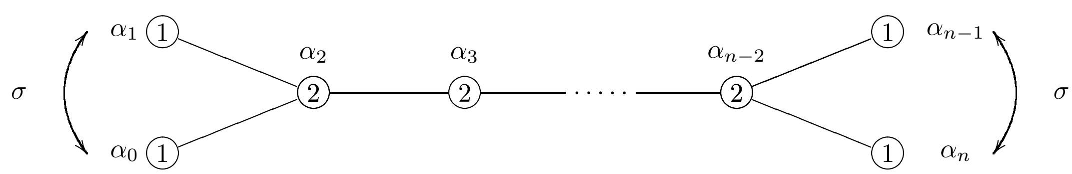
The action of σ on $\tilde{\Delta}$ is given by the above diagram : $\sigma(\alpha_0) = \alpha_1$, $\sigma(\alpha_1) = \alpha_0$, $\sigma(\alpha_i) = \alpha_i$ for every $i \in \{2, 3, \ldots, n\}$.

Let $\Omega_0 = \{\alpha_0\}$, $\Omega_1 = \{\alpha_0, \alpha_1\}$ and, for $2 \leqslant i \leqslant n$, let $\Omega_i = \{\alpha_i\}$. Then $\Omega_i \in \mathcal{Q}(\mathbf{G})$. Moreover, one can check that $\{\Omega_0, \Omega_1, \ldots, \Omega_n\}$ is a set of representatives of $\mathcal{A}_{\mathbf{G}}$-orbits in $\mathcal{Q}(\mathbf{G})$. Then

$$t_{\Omega_i}^2 = 1 \quad \text{and} \quad \dim \operatorname{Ker}(t_{\Omega_i} + \operatorname{Id}_V) = i.$$

This shows that $t_{\Omega_i} = t_i$: we retrieve Corollary 4.14.

4.E.2. Type D. We assume here that $\dim V = 2n$ and that $n \geqslant 4$. We set $\alpha_0 = -\tilde{\alpha}_1$. Then $\mathcal{A}_{\mathbf{G}}$ is of order 2. We denote by σ its unique non-trivial element. Note that $\mathcal{A}_{\mathbf{G}} \neq \mathcal{A}$. We number the affine Dynkin diagram of $\mathbf{G}$ as follows :



The natural number written inside the node α_i is equal to n_{α_i}. We have

$$\sigma = {}^{s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{n-2}}} (s_{\alpha_{n-1}} s_{\alpha_n}).$$

The action of σ on $\tilde{\Delta}$ is given in the above diagram : $\sigma(\alpha_0) = \alpha_1$, that $\sigma(\alpha_1) = \alpha_0$ and that $\sigma(\alpha_i) = \alpha_i$ for every $i \in \{2, 3, \ldots, n-2\}$, $\sigma(\alpha_{n-1}) = \alpha_n$ and $\sigma(\alpha_n) = \alpha_{n-1}$.

Let $\Omega_0 = \{\alpha_0\}$, $\Omega_1 = \{\alpha_0, \alpha_1\}$, $\Omega_i = \{\alpha_i\}$ (for $2 \leqslant i \leqslant n-2$), $\Omega_{n-1} = \{\alpha_{n-1}, \alpha_n\}$ and $\Omega_n = \{\alpha_n\}$. Then $\Omega_i \in \mathcal{Q}(\mathbf{G})$. Moreover, one can check that $\{\Omega_0, \Omega_1, \ldots, \Omega_n\}$ is a set of representatives of $\mathcal{A}_{\mathbf{G}}$-orbits in $\mathcal{Q}(\mathbf{G})$. Then

$$t_{\Omega_i}^2 = 1 \quad \text{and} \quad \dim \operatorname{Ker}(t_{\Omega_i} + \operatorname{Id}_V) = i.$$

This shows that $t_{\Omega_i} = t_i$: we retrieve Corollary 4.14.

5. Adjoint simple groups

The aim of this section is to provide complete tables for isolated and quasi-isolated semisimple conjugacy classes in adjoint simple groups. For classical groups, we also give a description in terms of their natural representation.

> **Hypothesis :** *In this section, and only in this section, we assume that* $\mathbf{G}$ *is adjoint and simple.*

The root system Φ is then irreducible. We denote by α_0 the root $-\tilde{\alpha}_1$. Note also that $\mathcal{A}_{\mathbf{G}} = \mathcal{A}$. Moreover, for every $\alpha \in \tilde{\Delta}$, we have $o_{\mathbf{G}}(\varpi_\alpha^\vee) = 1$. Therefore, the Theorem 4.6 can be stated as follows :

Theorem 5.1. *Assume that* $\mathbf{G}$ *is adjoint and simple. Then* $\mathcal{Q}(\mathbf{G})_{p'}$ *is the set of subsets* Ω *in* $\tilde{\Delta}_{p'}$ *which are acted on transitively by their stabilizer in* $\mathcal{A}$. *If* $\Omega \in \mathcal{Q}(\mathbf{G})_{p'}$, *let* n_Ω *denote the number* n_α *(for some* $\alpha \in \Omega$*). Then :*

$$t_\Omega = \tilde{\imath}_{\mathbf{T}}\Big(\frac{1}{n_\Omega . |\Omega|} \sum_{\alpha \in \Omega} \varpi_\alpha^\vee\Big).$$

We have :

(a) *The map* $\mathcal{Q}(\mathbf{G})_{p'} \to \mathbf{T}$, $\Omega \mapsto t_\Omega$ *induces a bijection between the set of orbits of* $\mathcal{A}_{p'}$ *in* $\mathcal{Q}(\mathbf{G})_{p'}$ *and the set of conjugacy classes of quasi-isolated semisimple elements in* $\mathbf{G}$.

(b) *Let* $\Omega \in \mathcal{Q}(\mathbf{G})_{p'}$. *Then :*
  (α) $W^\circ(t_\Omega) = W_{\tilde{\Delta}-\Omega}$;
  (β) $A_{\mathbf{G}}(t_\Omega) = \{z \in \mathcal{A}_{\mathbf{G}} \mid z(\Omega) = \Omega\}$;
  (γ) $o(t_\Omega) = n_\Omega |\Omega|$;
  (δ) s_Ω *is* $\mathbf{G}$*-isolated if and only if* $|\Omega| = 1$.

This implies that the set of conjugacy classes of isolated semisimple elements in $\mathbf{G}$ is in bijection with the set of orbits of $\mathcal{A}$ in $\tilde{\Delta}_{p'}$.

5.A. Classification by use of the affine Dynkin diagram. We first set some notation. We denote by α_0 the root $\tilde{\alpha}_1$ (recall that $r = 1$). Let n denote the rank of $\mathbf{G}$ (i.e. $n = |\Delta|$). We write $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. If $0 \leqslant i \leqslant n$, we set $n_{\alpha_i} = n_i$, $z_{\alpha_i} = z_i$ and $\varpi_{\alpha_i}^\vee = \varpi_i^\vee$. Note that $z_i(\alpha_0) = \alpha_i$. The Table I gives the list of all the affine Dynkin diagrams together with the structure of $\mathcal{A}$ (see [Bou, Planches I-IX]).

We give in Table II the classification of conjugacy classes of quasi-isolated elements in adjoint classical groups. In Table III, we deal with the adjoint groups of exceptional type E_6 and E_7. We have not included adjoint groups of type E_8, F_4 and G_2 since they are also simply connected. Therefore, Proposition 4.9, Theorem 5.1 and Table I gives easily all informations concerning the (quasi-)isolated elements for these groups.

5.B. Explicit descriptions for adjoint classical groups. The case of special orthogonal groups was done in subsection 4.E. Therefore, we only have to investigate adjoint classical groups of type A, C and D.

5.B.1. Type A. Assume here that $\tilde{\mathbf{G}} = \mathbf{GL}_{n+1}(\mathbb{F})$, that $\mathbf{G} = \mathbf{PGL}_{n+1}(\mathbb{F})$ and that $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$ is the canonical morphism (here, n is a non-zero natural number). Let I_{n+1} denote the identity matrix of $\mathbf{GL}_{n+1}(\mathbb{F})$. If d is a non-zero natural number invertible in $\mathbb{F}$, we denote by ζ_d a primitive d-th root of unity in $\mathbb{F}^\times$ and we set $J_d = \mathrm{diag}(1, \zeta_d, \zeta_d^2, \ldots, \zeta_d^{d-1}) \in \mathbf{GL}_d(\mathbb{F})$.

Now, let $\mathrm{Div}_{p'}(n+1)$ denote the set of divisors of $n+1$ which are invertible in $\mathbb{F}$. If $d \in \mathrm{Div}_{p'}(n+1)$, let $\tilde{s}_{n+1,d}$ denote the matrix $I_{\frac{n+1}{d}} \otimes J_d \in \tilde{\mathbf{G}}$. We set $s_{n+1,d} = \pi(\tilde{s}_{n+1,d})$.

Note that $\mathcal{A}_{p'}$ is cyclic of order $n_{p'} = |\tilde{\Delta}_{p'}|$ and that it acts transitively on $\tilde{\Delta}_{p'}$. If $d \in \mathrm{Div}_{p'}(n+1)$, we denote by $\Omega_{n+1,d}$ the orbit of α_0 under the unique subgroup of order d of $\mathcal{A}$: $\Omega_{n+1,d} = \{\alpha_{j(n+1)/d} \mid 0 \leqslant j \leqslant d-1\}$.

Proposition 5.2. *If $\mathbf{G} = \mathbf{PGL}_{n+1}(\mathbb{F})$, then the map $\mathrm{Div}_{p'}(n+1) \to \mathbf{G}$, $d \mapsto s_{n+1,d}$ is a bijection between $\mathrm{Div}_{p'}(n+1)$ and the set of conjugacy classes of quasi-isolated semisimple elements in $\mathbf{G}$. Through the parametrization of Theorem 5.1, this corresponds to the map $\mathrm{Div}_{p'}(n+1) \mapsto \mathcal{Q}(\mathbf{G})_{p'}$, $d \mapsto \Omega_{n+1,d}$.*

If $d \in \mathrm{Div}_{p'}(n+1)$, then $s_{n+1,d}$ has order d, $W^\circ(s) \simeq (\mathfrak{S}_{n+1/d})^d$ and $A(s) \simeq (\mathbb{Z}/d\mathbb{Z})$ acts on $W^\circ(s)$ by permutation of the components. Moreover, $s_{n+1,d}$ is isolated if and only if $d = 1$.

5.B.2. Type C. We assume here that $p \neq 2$. Let V be a $2n$-dimensional vector space over $\mathbb{F}$, with $n \geqslant 2$. Let $\beta : V \times V \to \mathbb{F}$ be a non-degenerate skew-symmetric bilinear form. We assume here that $\tilde{\mathbf{G}} = \mathbf{Sp}(V, \beta)$, that $\mathbf{G} = \tilde{\mathbf{G}}/\{\mathrm{Id}_V, -\mathrm{Id}_V\}$ and that $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$ is the canonical morphism.

Proposition 5.3. *Let $\tilde{s} \in \tilde{\mathbf{G}}$ be semisimple and let $s = \pi(\tilde{s})$.*

- (a) *If s is quasi-isolated, then $\tilde{s}^4 = 1$.*
- (b) *If $\tilde{s}^2 = 1$, then s is isolated.*
- (c) *If $\tilde{s}^4 = 1$ and $\tilde{s}^2 \neq 1$, then s is quasi-isolated if and only if $\dim \mathrm{Ker}(\tilde{s} - \mathrm{Id}_V) = \dim \mathrm{Ker}(\tilde{s} + \mathrm{Id}_V)$.*

Proof - (a) follows immediately from Corollary 2.11 and Example 4.10. (b) and (c) follow from direct computation. ■

If $0 \leqslant i \leqslant n/2$, let $\tilde{t}_i$ be an element of $\tilde{\mathbf{G}}$ such that $\tilde{t}_i^2 = 1$ and $\dim \mathrm{Ker}(\tilde{t}_i + \mathrm{Id}_V) = 2i$. If $0 \leqslant i < n/2$, let $\tilde{s}_i$ be an element of $\tilde{\mathbf{G}}$ of order 4 such that $\dim \mathrm{Ker}(\tilde{s}_i - \mathrm{Id}_V) = \dim \mathrm{Ker}(\tilde{s}_i + \mathrm{Id}_V) = i$. We set $t_i = \pi(\tilde{t}_i)$ and $s_i = \pi(\tilde{s}_i)$.

Corollary 5.4. *The set $\{t_i \mid 0 \leqslant i \leqslant n/2\} \cup \{s_i \mid 0 \leqslant i < n/2\}$ is a set of representatives of quasi-isolated elements of $\mathbf{G}$. The subset of $\tilde{\Delta}$ associated to t_i (respectively s_i) through the parametrization of Theorem 5.1 is $\{\alpha_i\}$ (respectively $\{\alpha_i, \alpha_{n-i}\}$).*

5.B.3. Type D. We assume here that $p \neq 2$. Let V be a $2n$-dimensional vector space over $\mathbb{F}$, with $n \geqslant 3$. Let $\beta : V \times V \to \mathbb{F}$ be a non-degenerate symmetric bilinear form. We assume here that $\tilde{\mathbf{G}} = \mathbf{SO}(V, \beta)$, that $\mathbf{G} = \tilde{\mathbf{G}}/\{\mathrm{Id}_V, -\mathrm{Id}_V\}$ and that $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$ is the canonical morphism.

Proposition 5.5. *Let $\tilde{s} \in \tilde{\mathbf{G}}$ be semisimple and let $s = \pi(\tilde{s})$.*

- (a) *If s is quasi-isolated, then $\tilde{s}^4 = 1$.*
- (b) *If $\tilde{s}^2 = 1$, then s is quasi-isolated. Moreover, s is isolated if and only if $\dim \mathrm{Ker}(\tilde{s} - \mathrm{Id}_V) \notin \{1, n-1\}$.*
- (c) *If $\tilde{s}^4 = 1$ and $\tilde{s}^2 \neq 1$, then s is quasi-isolated if and only if $\dim \mathrm{Ker}(\tilde{s} - \mathrm{Id}_V) = \dim \mathrm{Ker}(\tilde{s} + \mathrm{Id}_V)$ and $\dim \mathrm{Ker}(\tilde{s} - \mathrm{Id}_V) \neq 0$ if n is odd.*

Proof - (a) follows immediately from Corollary 2.11 and Proposition 5.5. (b) and (c) follow from direct computations. ■

If $0 \leqslant i \leqslant n/2$, let $\tilde{t}_i$ be an element of $\tilde{\mathbf{G}}$ such that $\tilde{t}_i^2 = 1$ and $\dim \mathrm{Ker}(\tilde{t}_i + \mathrm{Id}_V) = 2i$. If $1 \leqslant i < n/2$, let $\tilde{s}_i$ be an element of $\tilde{\mathbf{G}}$ of order 4 such that $\dim \mathrm{Ker}(\tilde{s}_i - \mathrm{Id}_V) = \dim \mathrm{Ker}(\tilde{s}_i + \mathrm{Id}_V) = i$. We set $t_i = \pi(\tilde{t}_i)$ and $s_i = \pi(\tilde{s}_i)$. Finally, there are two conjugacy classes of elements $\tilde{s}$ of order 4 such that $\dim \mathrm{Ker}(\tilde{s} - \mathrm{Id}_V) = \dim \mathrm{Ker}(\tilde{s} + \mathrm{Id}_V) = 0$ (these two conjugacy classes are in correspondence through the non trivial automorphism of the Dynkin diagram of $\mathbf{G}$) : we denote by $\tilde{s}_0$ and $\tilde{s}'_0$ some representatives of these two classes. We set $s_0 = \pi(\tilde{s}_0)$ and $s'_0 = \pi(\tilde{s}'_0)$. We set $E_n = \varnothing$ if n is odd and $E_n = \{s_0, s'_0\}$ if n is even.

Corollary 5.6. *The set $\{t_0\} \cup \{t_i \mid 2 \leqslant i \leqslant n/2\}$ is a set of representatives of isolated elements of* $\mathbf{G}$*. The subset of $\tilde{\Delta}$ associated to t_i through the parametrization of Theorem 5.1 is $\{\alpha_i\}$.*

The set $\{t_1\} \cup \{s_i \mid 1 \leqslant i < n/2\} \cup E_n$ is a set of representatives of quasi-isolated but non-isolated elements of $\mathbf{G}$*. Through the parametrization of Theorem 5.1, t_1 is associated to $\{\alpha_0, \alpha_1\}$, s_1 is associated to $\{\alpha_0, \alpha_1, \alpha_{n-1}, \alpha_n\}$ and s_i is associated to $\{\alpha_i, \alpha_{n-i}\}$. If n is even, s_0 and s_0' correspond to $\{\alpha_0, \alpha_{n-1}\}$ and $\{\alpha_0, \alpha_n\}$ (or conversely).*

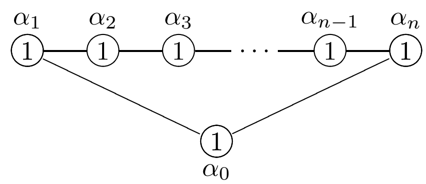
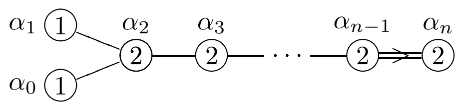
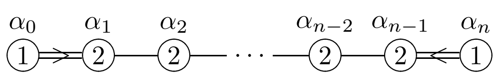
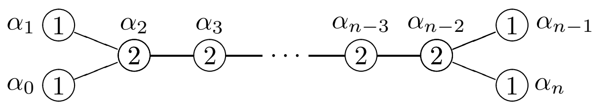
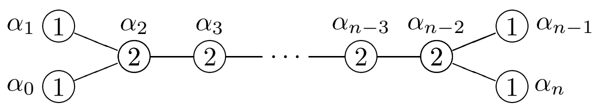
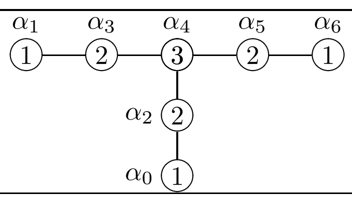
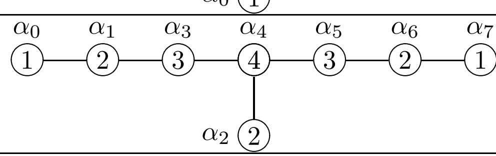
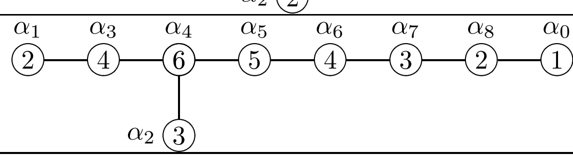
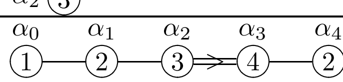
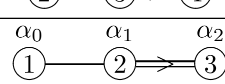
Type of $\mathbf{G}$	$\tilde{\Delta}$	$\mathcal{A}$	$\lvert\mathcal{A}\rvert$
A_n		$< z_1 >$	$n+1$
B_n		$< z_1 >$	2
C_n		$< z_n >$	2
D_n n even		$< z_1 > \times < z_n >$	4
D_n n odd		$< z_n >$	4
E_6		$< z_1 >$	3
E_7		$< z_7 >$	2
E_8		1	1
F_4		1	1
G_2		1	1

Table I. Affine Dynkin diagrams

$\mathbf{G}$	Ω	p ?	$o(s_\Omega)$	$C^\circ_{\mathbf{G}}(s_\Omega)$	$\lvert A(s_\Omega)\rvert$	isolated ?
A_n	$\{\alpha_{j(n+1)/d} \mid 0 \leqslant j \leqslant d-1\}$ for $d \mid n+1$	$p \nmid d$	d	$(A_{(n+1)/d}-1)^d$	d	iff $d=1$
B_n	$\{\alpha_0\}$		1	B_n	1	yes
	$\{\alpha_0, \alpha_1\}$	$p \neq 2$	2	B_{n-1}	2	no
	$\{\alpha_d\},\ 2 \leqslant d \leqslant n$	$p \neq 2$	2	$D_d \times B_{n-d}$	2	yes
C_n	$\{\alpha_0\}$		1	C_n	1	yes
	$\{\alpha_d\},\ 1 \leqslant d < n/2$	$p \neq 2$	2	$C_d \times C_{n-d}$	1	yes
	$\{\alpha_{n/2}\}$ (if n is even)	$p \neq 2$	2	$C_{n/2} \times C_{n/2}$	2	yes
	$\{\alpha_0, \alpha_n\}$	$p \neq 2$	2	A_{n-1}	2	no
	$\{\alpha_d, \alpha_{n-d}\},\ 1 \leqslant d < n/2$	$p \neq 2$	4	$(B_d)^2 \times A_{n-2d-1}$	2	no
D_n	$\{\alpha_0\}$		1	D_n	1	yes
	$\{\alpha_d\},\ 2 \leqslant d < n/2$	$p \neq 2$	2	$D_d \times D_{n-d}$	2	yes
	$\{\alpha_{n/2}\}$ (if n is even)	$p \neq 2$	4	$D_{n/2} \times D_{n/2}$	4	yes
	$\{\alpha_d, \alpha_{n-d}\},\ 2 \leqslant d < n/2$	$p \neq 2$	4	$(D_d)^2 \times A_{n-2d-1}$	4	no
	$\{\alpha_0, \alpha_1, \alpha_{n-1}, \alpha_n\}$	$p \neq 2$	4	A_{n-3}	4	no
	$\{\alpha_0, \alpha_1\}$	$p \neq 2$	2	D_{n-1}	2	no
	$\{\alpha_0, \alpha_{n-1}\}$ (if n is even)	$p \neq 2$	2	A_{n-1}	2	no
	$\{\alpha_0, \alpha_n\}$ (if n is even)	$p \neq 2$	2	A_{n-1}	2	no

Table II. Quasi-isolated elements in adjoint classical groups

$\mathbf{G}$	Ω	p ?	$o(s_\Omega)$	$C^\circ_{\mathbf{G}}(s_\Omega)$	$\lvert A(s_\Omega)\rvert$	isolated ?
E_6	$\{\alpha_0\}$		1	E_6	1	yes
	$\{\alpha_2\}$	$p \neq 2$	2	$A_5 \times A_1$	1	yes
	$\{\alpha_4\}$	$p \neq 3$	3	$A_2 \times A_2 \times A_2$	3	yes
	$\{\alpha_0, \alpha_1, \alpha_6\}$	$p \neq 3$	3	D_4	3	no
	$\{\alpha_2, \alpha_3, \alpha_5\}$	$p \notin \{2, 3\}$	6	$A_1 \times A_1 \times A_1 \times A_1$	3	no
E_7	$\{\alpha_0\}$		1	E_7	1	yes
	$\{\alpha_1\}$	$p \neq 2$	2	$A_1 \times D_6$	1	yes
	$\{\alpha_2\}$	$p \neq 2$	2	A_7	2	yes
	$\{\alpha_3\}$	$p \neq 3$	3	$A_2 \times A_5$	1	yes
	$\{\alpha_4\}$	$p \neq 2$	4	$A_3 \times A_3 \times A_1$	2	yes
	$\{\alpha_0, \alpha_7\}$	$p \neq 2$	2	E_6	2	no
	$\{\alpha_1, \alpha_6\}$	$p \neq 2$	4	$D_4 \times A_1 \times A_1$	2	no
	$\{\alpha_3, \alpha_5\}$	$p \notin \{2, 3\}$	6	$A_2 \times A_2 \times A_2$	2	no

TABLE III. QUASI-ISOLATED ELEMENTS IN ADJOINT GROUPS OF TYPE E_6 AND E_7

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